Sparling two-forms, the conformal factor and the the gravitational energy density of the teleparallel equivalent of general relativity

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Abstract

It has been shown recently that within the framework of the teleparallel equivalent of general relativity (TEGR) it is possible to define in an unequivocal way the energy density of the gravitational field. The TEGR amounts to an *alternative formulation* of Einstein's general relativity, not to an alternative gravity theory. The localizability of the gravitational energy has been investigated in a number of spacetimes with distinct topologies, and the outcome of these analises agree with previously known results regarding the exact expression of the gravitational energy, and/or with the specific properties of the spacetime manifold. In this article we establish a relationship between the expression for the gravitational energy density of the TEGR and the Sparling two-forms, which are known to be closely connected with the gravitational energy. We also show that our expression of energy yields the correct value of gravitational mass contained in the conformal factor of the metric field.

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I. Introduction

The problem of a consistent definition of the gravitational energy has been addressed over the years by means of rather different methods. The traditional approach amounts to defining the total gravitational energy of an asymptotically flat spacetime through the use of pseudotensors. A closely related approach consists in associating the gravitational energy with the surface term that appears in the total Hamiltonian of the gravitational field. An alternative way of defining the gravitational energy is provided by the concept of quasi-local energy. The quasi-local definition of energy, momentum and angular momentum associates these quantities with an arbitrary two-surface S of an arbitrary spacetime manifold. This approach is presented in ref.[1], where a comprehensive list of references on the problem of energy in general relativity is presented.

A recent approach to the problem of the definition of the gravitational energy has arisen in the framework of the teleparallel equivalent of general relativity (TEGR)[2, 3, 4]. The latter is an alternative formulation of general relativity. This formulation is established by means of the tetrad field $e_{a\mu}$ and the spin connection $\omega_{\mu ab}$, which are totally independent field quantities, even at the level of field equations. The metric tensor obeys Einstein's equations. However, the action integral is constructed in terms of the torsion tensor. The Lagrangian density of the TEGR is a functional of the torsion tensor, but is precisely equivalent to the ordinary scalar density $eR(e_{a\mu})$, provided the curvature tensor $R_{ab\mu\nu}(\omega)$ vanishes. In fact the vanishing of $R_{ab\mu\nu}(\omega)$ is one basic motivation for considering the TEGR, since this property leads to the establishment of a *reference space*[5].

The definition of the gravitational energy *density* emerges in the context of the Hamiltonian formulation of the TEGR. The latter is considered in ref.[3]. It has been shown that, under a suitable gauge fixing, the Hamiltonian formulation of the TEGR is well defined, as the constraints turn out to be first class. The major property of the Hamiltonian formulation is that the Hamiltonian constraint $C = 0$ can be written as

$$C \;=\; H \;-\; E_{ADM} \;=\; 0\;,$$

in the case of asymptotically flat spacetimes[2, 4]. H is interpreted as the effective Hamiltonian. The Arnowitt-Deser-Misner (ADM) energy[6] (E_{ADM})

arises in the Hamiltonian constraint upon integration over the whole three-dimensional spacelike hypersurface of a scalar density $\varepsilon(x)$, which can be written as a total divergence of the form $\frac{1}{8\pi G}\partial_i(eT^i)$, where T^i is the trace of the torsion tensor. We propose that $\varepsilon(x)$ represents the gravitational energy density for spacetimes with any topology. In fact we have applied it to the determination of the distribution of gravitational energy in de Sitter space and found that the resulting analysis is in total agreement with the phenomenological features of this space[7]. Therefore we assert that the Hamiltonian constraint equation can be written as $C = H - E = 0$ for *any* spacetime.

The existence of the gravitational energy density as given above allows the conclusion that the gravitational energy is localizable, in the same way as the electromagnetic energy is also localizable. The very idea of a black hole lends support to this conclusion. If there exists an ammount of gravitational energy (mass) inside the event horizon of a black hole, then by means of a coordinate transformation we do not expect to make this energy vanish.

In this article we establish a relationship between the gravitational energy density $\varepsilon(x)$ and the Sparling two-forms. The latter are known to yield the gravitational energy and momentum of an asymptotically flat spacetime by means of an integration over a two-surface at infinity. In fact the Sparling forms are closely connected with various pseudotensors of energy-momentum of the gravitational field. We also show that under integration $\varepsilon(x)$ captures the information about the gravitational mass contained in the conformal factor of the metric field, provided the appropriate boundary conditions at infinity are imposed. Thus our expression of energy encompasses the features of distinct approaches to the definition of the gravitational energy.

We remark that although there exist in the literature several expressions for the definition of the *total* gravitational energy, the present expression is unique as it is the only one that possesses the feature of localizability. It is precisely this feature that allows obtaining the striking result regarding rotating black holes (cf. ref.[5]), namely, that the energy contained within the outer horizon of a Kerr black hole is essentially equal to twice its irreducible mass.

Notation: spacetime indices $\mu, \nu, ...$ and local Lorentz indices $a, b, ...$ run from 0 to 3. In the 3+1 decomposition latin indices from the middle of the alphabet indicate space indices according to $\mu = 0, i, \quad a = (0), (i)$. The tetrad field $e^a\,_\mu$ and the spin connection $\omega_{\mu ab}$ yield the usual definitions

of the torsion and curvature tensors: $R^a{}_{b\mu\nu} = \partial_\mu \omega_\nu{}^a{}_b + \omega_\mu{}^a{}_c \omega_\nu{}^c{}_b - ...$, $T^a{}_{\mu\nu} = \partial_\mu e^a{}_\nu + \omega_\mu{}^a{}_b e^b{}_\nu - ....$ The flat spacetime metric is fixed by $\eta_{(0)(0)} = -1$.

II. The TEGR

Throughout the paper we will consider only asymptotically flat spacetimes, because it is in this geometrical framework that the Sparling two-forms are relevant for energy considerations. The Lagrangian density of the TEGR in empty spacetime is given by

$$L(e, \omega, \lambda) = -ke(\frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{bac} - T^a T_a) + e\lambda^{ab\mu\nu} R_{ab\mu\nu}(\omega) . \quad (1)$$

where $k = \frac{1}{16\pi G}$, G is the gravitational constant; $e = det(e^a{}_\mu)$, $\lambda^{ab\mu\nu}$ are Lagrange multipliers and T_a is the trace of the torsion tensor defined by $T_a = T^b{}_{ba}$. The tetrad field $e_{a\mu}$ and the spin connection $\omega_{\mu ab}$ are completely independent field variables. The latter is enforced to satisfy the condition of zero curvature. Therefore this Lagrangian formulation is in no way similar to the usual Palatini formulation, in which the spin connection is related to the tetrad field via field equations.

The equivalence of the TEGR with Einstein's general relativity is based on the identity

$$eR(e, \omega) = eR(e) + e(\frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{bac} - T^a T_a) - 2\partial_\mu(eT^\mu) , \quad (2)$$

which is obtained by just substituting the arbitrary spin connection $\omega_{\mu ab} \equiv {}^o\omega_{\mu ab}(e) + K_{\mu ab}$ in the scalar curvature tensor $R(e, \omega)$ in the left hand side; ${}^o\omega_{\mu ab}(e)$ is the Levi-Civita connection and $K_{\mu ab} = \frac{1}{2}e_a{}^\lambda e_b{}^\nu(T_{\lambda\mu\nu} + T_{\nu\lambda\mu} - T_{\mu\nu\lambda})$ is the contorsion tensor. The vanishing of $R^a{}_{b\mu\nu}(\omega)$, which is one of the field equations derived from (1), implies the equivalence between the scalar curvature $R(e)$, constructed out of $e^a{}_\mu$ only, and the quadratic combination of the torsion tensor. It also ensures that the field equations arising from the variation of L with respect to $e^a{}_\mu$ are strictly equivalent to Einstein's

equations in tetrad form. Let $\frac{\delta L}{\delta e^{a\mu}} = 0$ denote the field equations satisfied by $e^{a\mu}$. It can be shown by explicit calculations that

$$\frac{\delta L}{\delta e^{a\mu}} = \frac{1}{2}\{R_{a\mu}(e) - \frac{1}{2}e_{a\mu}R(e)\} . \tag{3}$$

We refer the reader to refs.[3, 4] for additional details.

In the Lagrangian density (1) we have not included the total divergence $\partial_\mu(eT^\mu)$. The reason is that the variation of the latter in the action integral causes the appearance, via integration by parts, of surface terms that do not vanish when $r \to \infty$. In this limit we should consider variations in $g_{\mu\nu}$ or in $e_{a\mu}$ that preserve the asymptotic structure of the flat spacetime metric; the allowed coordinate transformations must be of the Poincaré type. The variation of $\partial_\mu(eT^\mu)$ at infinity under such variations of $e_{a\mu}$ does not vanish. Therefore the total divergence has to be dropped down. On the other hand, all surface terms arising from partial integration in the variation of the action integral constructed out of (1) vanish in the limit $r \to \infty$. The action based solely on the quadratic torsion terms does not require additional surface terms, as it is invariant under transformations that preserve the asymptotic structure of the field quantities. This property fixes the action integral, together with the requirement that the variation of the latter must yield Einstein's equations, which is actually the case in view of (3) (the Hilbert-Einstein Lagrangian requires the addition of a surface term for the variation of the action to be well defined; a clear discussion of this point is given in ref.[8]). Unfortunately in refs.[2-5] we have not given the correct argument for the suppression of $\partial_\mu(eT^\mu)$. We argued, erroneously, that the variation of this divergence vanishes at infinity when we consider variations of $e_{a\mu}$ as defined above. This divergence has to be subtracted by hand, in the same way that the Hilbert-Einstein Lagrangian density requires an additional surface term.

In what follows we will be interested in asymptoticaly flat spacetimes. The Hamiltonian formulation of the TEGR can be successfully implemented if we fix the gauge $\omega_{0ab} = 0$ from the outset, since in this case the constraints (to be shown below) constitute a *first class* set[3]. The condition $\omega_{0ab} = 0$ is achieved by breaking the local Lorentz symmetry of (1). We still make use of the residual time independent gauge symmetry to fix the usual time gauge condition $e_{(k)}\,^0 = e_{(0)i} = 0$. Because of $\omega_{0ab} = 0$, H does not depend on P^{kab}, the momentum canonically conjugated to ω_{kab}. Therefore arbitrary variations of $L = p\dot{q} - H$ with respect to P^{kab} yields $\dot{\omega}_{kab} = 0$. Thus in view

of $\omega_{0ab}=0$, ω_{kab} drops out from our considerations. The above gauge fixing can be understood as the fixation of a *global* reference frame.

As a consequence of the above gauge fixing the canonical action integral obtained from (1) becomes[4]

$$A_{TL} \;=\; \int d^4x\{\Pi^{(j)k}\dot{e}_{(j)k} \;-\; H\}\;, \tag{4}$$

$$H \;=\; NC \;+\; N^iC_i \;+\; \Sigma_{mn}\Pi^{mn} \;+\; \frac{1}{8\pi G}\partial_k(NeT^k) \;+\; \partial_k(\Pi^{jk}N_j)\;. \tag{5}$$

N and N^i are the lapse and shift functions, and $\Sigma_{mn}=-\Sigma_{nm}$ are Lagrange multipliers. The constraints are defined by

$$C \;=\; \partial_j(2keT^j) \;-\; ke\Sigma^{kij}T_{kij} \;-\; \frac{1}{4ke}(\Pi^{ij}\Pi_{ji}-\frac{1}{2}\Pi^2)\;, \tag{6}$$

$$C_k \;=\; -e_{(j)k}\partial_i\Pi^{(j)i} \;-\; \Pi^{(j)i}T_{(j)ik}\;, \tag{7}$$

with $e=det(e_{(j)k})$ and $T^i \;=\; g^{ik}e^{(j)l}T_{(j)lk}$. We remark that (4) and (5) are invariant under global SO(3) and general coordinate transformations.

We assume the asymptotic behaviour $e_{(j)k}\approx\eta_{jk}+\frac{1}{2}h_{jk}(\frac{1}{r})$ for $r\rightarrow\infty$. In view of the relation

$$\frac{1}{8\pi G}\int d^3x\partial_j(eT^j) \;=\; \frac{1}{16\pi G}\int_S dS_k(\partial_i h_{ik}-\partial_k h_{ii}) \;\equiv\; E_{ADM} \tag{8}$$

where the surface integral is evaluated for $r\rightarrow\infty$, the integral form of the Hamiltonian constraint $C=0$ may be rewritten as

$$\int d^3x\left\{ke\Sigma^{kij}T_{kij}+\frac{1}{4ke}(\Pi^{ij}\Pi_{ji}-\frac{1}{2}\Pi^2)\right\} \;=\; E_{ADM}\;. \tag{9}$$

The integration is over the whole three dimensional space. Given that $\partial_j(eT^j)$ is a scalar density, from (7) and (8) we define the gravitational energy density enclosed by a volume V of the space as

$$E_g \;=\; \frac{1}{8\pi G}\int_V d^3x\partial_j(eT^j)\;. \tag{10}$$

It must be noted that E_g depends only on the triads $e_{(k)i}$ restricted to a three-dimensional spacelike hypersurface; the inverse quantities $e^{(k)i}$ can be written in terms of $e_{(k)i}$. From the identity (3) we observe that the dynamics of the triads does not depend on $\omega_{\mu ab}$. Therefore E_g given above does not depend on the fixation of any gauge for $\omega_{\mu ab}$.

The reference space which defines the zero of energy has been discussed in ref.[5]. Here we will briefly present the main ideas about its fixation. The establishment of a reference space requires the concepts of holonomic and anholonomic transformations of coordinates. Let us consider the Euclidean space with metric $\eta_{(i)(j)} = (+++)$, which is the spatial section of the Minkowski metric. We introduce coordinates $q^{(i)}$ such that the line element of the Euclidean space is written as $ds^2 = \eta_{(i)(j)} dq^{(i)} dq^{(j)}$. We consider next a coordinate transformations $dq^{(i)} = e^{(i)}{}_j(x) dx^j$, which allows us to rewrite $ds^2 = \eta_{(i)(j)} e^{(i)}{}_m(x) e^{(j)}{}_n(x) dx^m dx^n = g_{mn} dx^m dx^n$. This transformation can be holonomic or anholonomic.

If the relation $dq^{(i)} = e^{(i)}{}_j(x) dx^j$ can be integrated over the whole three-dimensional space, the transformation $q^{(i)} \to x^j$ corresponds to a single-valued global transformation, and therefore it is called holonomic. Both sets of coordinates, $\{q^{(i)}\}$ and $\{x^j\}$, describe the three-dimensional Euclidean space, and we have necessarily that $e^{(i)}{}_j$ is a gradient vector, i.e., $e^{(i)}{}_j = \frac{\partial q^{(i)}}{\partial x^j}$. However, in general $dq^{(i)} = e^{(i)}{}_j dx^j$ cannot be globally integrated, since $e^{(i)}{}_j$ may not be written as the gradient of a function, namely, $e^{(i)}{}_j$ may not be of the type $e^{(i)}{}_j = \partial_j q^{(i)}$. If the quantities $e^{(i)}{}_j$ are such that $\partial_j e^{(i)}{}_k - \partial_k e^{(i)}{}_j \neq 0$, then the transformation is called anholonomic.

For triads which are gradient vectors, the torsion tensor $T_{(i)jk} = \partial_j e_{(i)k} - \partial_k e_{(i)j}$ vanishes identically. A crucial result is that $T_{(i)jk}$ vanishes if and only if $\{e^{(i)}{}_j\}$ are gradient vectors[9]. In the framework of the Hamiltonian formulation of the TEGR the gravitational field corresponds to a configuration for which $T_{(i)jk} \neq 0$. We conclude that *every* gravitational field is *anholonomically* related to the three-dimensional Euclidean space, which is to be taken as the reference space. Since the torsion tensor vanishes for the latter, as we have seen, the gravitational energy of the reference space is zero.

III. The Sparling two-forms and its relation with the TEGR

The Sparling two-forms σ_a are defined by[10]

$$\sigma_a \;=\; -\frac{1}{2}\varepsilon_{abcd}\,\Gamma^{bc}\wedge e^d \;, \tag{11}$$

where ε_{abcd} is the totally antisymmetric Levi-Civita tensor such that $\varepsilon_{(0)(1)(2)(3)} = 1$; Γ^{ab} and e^a are one-forms, $\Gamma^{ab} = \Gamma_\mu{}^{ab}dx^\mu$, $e^a = e^a{}_\mu dx^\mu$, which are related by

$$de^a \;+\; \Gamma^a{}_b \wedge e^b \;=\; 0\,. \tag{12}$$

In components, $\Gamma_{\mu ab}$ turns out to be the Levi-Civita connection:

$$\Gamma_{\mu ab} \;=\; -\frac{1}{2}e^c{}_\mu(\Omega_{abc} \;-\; \Omega_{bac} \;-\; \Omega_{cab})$$

$$\Omega_{abc} \;=\; e_{a\lambda}(e_b{}^\nu\partial_\nu e_c{}^\lambda - e_c{}^\nu\partial_\nu e_b{}^\lambda)\;.$$

It is known that in a coordinate basis which is asymptotically cartesian the Sparling forms are related to various pseudotensors[11, 12]. In particular, it allows a definition of the *total* energy-momentum P_a of the gravitational field[12]:

$$P_a \;=\; -\frac{1}{8\pi G}\oint_{\partial\Sigma}\sigma_a \;, \tag{13}$$

where $\partial\Sigma$ actually represents a spacelike surface S at infinity (ω^{ab} of ref.[12] differs by an overall sign from Γ^{ab}). We mention that the connection between the Sparling forms with the Brown-York expression for quasi-local energy has been investigated in ref.[13].

In order to establish the relation between $P_{(0)}$ given by (13) and expression (10) we need to rewrite $\sigma_{(0)}$ in components. We have:

$$\begin{aligned}
\sigma_{(0)} \;&=\; -\frac{1}{2}\varepsilon_{(0)bcd}\Gamma^{bc}\wedge e^d \\
&=\; -\frac{1}{2}\varepsilon_{(i)(j)(k)}\Gamma_\mu{}^{(i)(j)}\,e^{(k)}{}_\nu dx^\mu\wedge dx^\nu \\
&=\; -\frac{1}{2}\varepsilon_{(i)(j)(k)}\Gamma_m{}^{(i)(j)}e^{(k)}{}_n dx^m\wedge dx^n\;.
\end{aligned}$$

The last equality is obtained in view of the fact that $\sigma^{(0)}$ is integrated over a spacelike surface.

Let us introduce the surface element dS_i defined by

$$dS_i \;=\; \frac{1}{2}\varepsilon_{ijk} dx^j \wedge dx^k \;,$$

where $\varepsilon^{123} = 1$. In view of the relations

$$e^{(i)}{}_m e^{(j)}{}_n e^{(k)}{}_l \,\varepsilon^{mnl} \;=\; e\,\varepsilon^{(i)(j)(k)}$$

$$e^{(k)}{}_m \,\varepsilon^{mnl} \;=\; e\, e_{(i)}{}^n e_{(j)}{}^l \varepsilon^{(k)(i)(j)} \;,$$

we find that $\sigma_{(0)}$ can be rewritten, after a number of manipulations, as

$$\sigma_{(0)} \;=\; -e\, e^{(i)m} e^{(j)n}\, \Gamma_{n(i)(j)} dS_m \;. \tag{14}$$

Therefore substitution of (14) in (13) leads to

$$P_{(0)} \;=\; \frac{1}{8\pi G} \oint_S e^{(i)m} e^{(j)n} \Gamma_{n(i)(j)} dS_m \;=\; \frac{1}{8\pi G} \int_V \partial_m (e\, e^{(i)m} e^{(j)n} \Gamma_{n(i)(j)}) d^3x \;. \tag{15}$$

It is not difficult to verify that if we assume the asymptotic behaviour $e_{(i)j} \simeq \eta_{ij} + \frac{1}{2} h_{ij}(\frac{1}{r})$ when $r \to \infty$, and impose the usual time gauge condition $e_{(i)}{}^0 = e^{(0)}{}_j = 0$ (the latter are *tetrad* components of the four-dimensional spacetime), then the expression above yields the ADM energy.

The relation of (15) with the energy expression (10) can now be established in a straightforward way. The equivalence between the two expressions rests on the identity

$$\partial_k (e\, e^{(i)k}\, e^{(j)n} \Gamma_{n(i)(j)}) \;\equiv\; \partial_k (e\, T^k) \;, \tag{16}$$

which can be verified by just substituting $\Gamma_{m(i)(j)}$ on the left hand side of the equation above. Note that because of the time gauge condition, which is also imposed in the Hamiltonian formulation of the TEGR, $\Gamma_{m(i)(j)}$ is constructed out of the triads components restricted to the three-dimensional spacelike hypersurface.

The equivalence between (10) and (15) can, however, be established only if the integration is performed over the whole three-dimensional space. The reason is that σ_a defined by (11) is normally considered a non-invariant quantity, as it transforms inhomogeneously under $e^a{}_\mu(x) \to \tilde{e}^a{}_\mu(x) = \Lambda^a{}_b(x)\, e^b{}_\mu(x)$,

where $\Lambda^a{}_b(x)$ belongs to the *local* $SO(3,1)$. On the other hand, in the framework of the TEGR $\partial_i(e\,T^i)$ is a scalar density, invariant under *global* $SO(3)$ transformations. This is a necessary requirement in order to arrive at a Hamiltonian formulation with only first class constraints.

IV. The conformal factor of the metric

An alternative way of defining the energy of an asymptotically flat gravitational field is by identifying it with the $O(r^{-1})$ part of the conformal factor of the metric[14]. However, this identification is only possible if some boundary contitions are imposed. Suppose that the metric field satisfies the following conditions in the asymptotic limit $r \to \infty$:

$$g_{ij} = \psi^4\,\delta_{ij} + \tilde{h}_{ij}\ ;\ \psi = 1 + O(r^{-1})\ ,$$

$$\tilde{h}_{ij} = O(r^{-1})\ ;\ tr_\delta\,\tilde{h}_{ij} = O(r^{-2})\ ;\ \partial_j \tilde{h}_{ij} = O(r^{-3})\ . \tag{17}$$

If this asymptotic behaviour is verified, the following definition is suggested for the energy of the gravitational field[14]:

$$E \;=\; -\frac{1}{2\pi G}\oint_{S\to\infty} dS_i\,\partial_i\psi\ . \tag{18}$$

We will show that a similar statement can be made in the context of expression (10). Let us consider the conformal factor ψ as above, with the asymptotic behaviour $\psi = 1 + O(r^{-1})$, and write

$$e_{(k)i} \;=\; \psi^2\,{}^o e_{(k)i} \,+\, \tilde{e}_{(k)i}\ , \tag{19}$$

where $\{{}^o e_{(k)i}\}$ are triads of the flat three-dimensional space (in cartesian coordinates, ${}^o e_{(k)i} = \delta_{ki}$) and $\tilde{e}_{(k)i}$ is such that for $r \to \infty$ we have $\tilde{e}_{(k)i} = O(r^{-1})$. In terms of these quantities we can construct the trace of the torsion tensor $T^i = g^{ik}\,e^{(l)j}\,T_{(l)jk}$:

$$T^i \;=\; g^{ik}\,e^{(l)j}[{}^o e_{(l)k}\,\partial_j\psi^2 - {}^o\,e_{(l)j}\,\partial_k\psi^2 + \partial_j\tilde{e}_{(l)k} - \partial_k\tilde{e}_{(l)j}]\ .$$

In order to evaluate (10) and compare it with expression (18) we will have to obtain the value of eT^i on a surface S at infinity. In this limit, the last two terms of T^i of the expression above contribute to the surface integral as

$$e^{(l)j}\,\partial_j\tilde{e}_{(l)k} \rightarrow \partial_j(\,^o e^{(l)j}\,\tilde{e}_{(l)k})\;, \tag{20}$$

$$e^{(l)j}\,\partial_k\tilde{e}_{(l)j} \rightarrow \partial_k(\,^o e^{(l)j}\tilde{e}_{(l)j})\;. \tag{21}$$

We would like to have these quantities vanishing under integration. For this purpose it is necessary to have them falling off as $\frac{1}{r^3}$ at infinity. This will be the case if we require

$$\partial_j(\,^o e^{(l)j}\,e_{(l)k}) = O(r^{-3})\;;\;^o e^{(l)j}\tilde{e}_{(l)j} = O(r^{-2})\;. \tag{22}$$

We observe that conditions (22) are the equivalent in triad form of conditions (17): the first condition above is equivalent to $\partial_j\tilde{h}_{ij} = O(r^{-3})$, the second one to $tr_\delta\,\tilde{h}_{ij} = O(r^{-2})$. Therefore we assume them to hold, together with (19), for asymptotically flat gravitational fields. As a consequence we obtain

$$\frac{1}{8\pi G}\oint_{S\rightarrow\infty} dS_i\;eT^i \;=\; -\frac{1}{2\pi G}\oint_{S\rightarrow\infty} dS_i\partial_i\,\psi\;, \tag{23}$$

which establishes the equivalence with (18).

As a simple application of expression (23) let us consider the Schwarzschild metric in isotropic coordinates. The metric for the spacelike hypersurface is given by

$$ds^2 \;=\; \left(1+\frac{m}{2r}\right)^4(dr^2 + r^2\,d\theta^2 + r^2\,sin^2\theta\,d\phi^2)\;, \tag{24}$$

where, for convenience, we make use of spherical coordinates. We have also made $G=1$. The triads associated with this metric read

$$e_{(k)i} \;=\; \begin{pmatrix} \psi^2\,sin\theta cos\phi & r\,\psi^2\,cos\theta cos\phi & -r\,\psi^2\,sin\theta sin\phi \\ \psi^2\,sin\theta sin\phi & r\,\psi^2\,cos\theta sin\phi & r\,\psi^2\,sin\theta cos\phi \\ \psi^2\,cos\theta & -r\,\psi^2\,sin\theta & 0 \end{pmatrix} \tag{25}$$

where $\psi = (1+\frac{m}{2r})$. In the expression above (k) and i are line and column indexes, respectively. It is easy to verify that $e^{(k)}\,_i e_{(k)j}$ yields precisely the metric components of the line element (24). We will exempt from presenting the details of the calculations, which in fact are not complicated. The only contribution to the surface integral is given by

$$eT^1 \;=\; eg^{1j}\,e^{(k)i}\,T_{(k)ij} \;=\; -4\,r^2\,sin\theta\,\psi\frac{\partial\psi}{\partial r}\;.$$

We find (recall that $G = 1$)

$$E \;=\; \frac{1}{8\pi}\oint_{S\to\infty} dS\,eT^1 \;=\; \frac{1}{8\pi}\int d\theta\,d\phi\,sin\theta\,2m \;=\; m\;, \tag{26}$$

as expected.

V. Comments

We have seen that expression (10) encompasses the features of previous, distinct approaches to the definition of the gravitational energy. Furthermore, it is still possible to consider *localized* gravitational energy. In this paper we have considered the definitions of gravitational energy (i) constructed out of the Sparling forms, and (ii) constructed out of the conformal factor of the metric field. None of these approaches arise in a natural or conventional way in the Lagrangian or Hamiltonian formulations of general relativity. Moreover, they provide only the *total* gravitational energy of asymptotically flat fields. We have seen that these definitions are in fact related to the definition (10) for the gravitational energy, which in turn does appear as one element of the Hamiltonian constraint C in the canonical formulation of the TEGR. Thus expressions (13) and (18) can be understood as different manifestations of the energy expression (10), when the integration is performed over the whole three-dimensional space. This result supports the general validity of expression (10).

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References

[1] J. D. Brown and J. W. York, Jr., Phys. Rev. **D47**, 1407 (1993).

[2] J. W. Maluf, Gen. Rel. Grav. **28**, 1361 (1996).

[3] J. W. Maluf, J. Math. Phys. **35**. 335 (1994).

[4] J. W. Maluf, J. Math. Phys. **36**, 4242 (1995).

[5] J. W. Maluf, E. F. Martins and A. Kneip, J. Math. Phys. **37**, 6302 (1996).

[6] R. Arnowitt, S. Deser and C. W. Misner, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962).

[7] J. W. Maluf, J. Math. Phys. **37**, 6293 (1996).

[8] L. D. Faddeev, Sov. Phys. Usp. **25**, 130 (1982).

[9] J. A. Schouten, *Tensor Analysis for Physicists*, 2^{nd} Edition (Dover, New York, 1989), page 82.

[10] G. A. Sparling, *Twistors, Spinors and the Einstein vacuum equations*, University of Pittsburgh preprint (1984); M. Dubois-Violette and J. Madore, Commun. Math. Phys. **108**, 213 (1987).

[11] R. Wallner and W. Thirring, Rev. Bras. Fis. (Brazil) **8**, 686 (1978).

[12] J. N. Goldberg, Phys. Rev. **D37**, 2116 (1988).

[13] S. Lau, Class. Quantum Grav. **10**, 2379 (1993).

[14] D. Brill, Ann. Phys. (N.Y.) **7**, 466 (1959); N. $\bar{O}$ Murchadha and J. W. York, Jr., Phys. Rev. **D10**, 2345 (1974); J. W. York, Jr., *Energy and Momentum of the Gravitational Field*, in *Essays in General Relativity*, Ed. F. Tipler (Academic Press, New York, 1980).